Topological Entropy: A New Principle from Worldline Non-Injectivity

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Abstract

We show that worldline non-injectivity — the kinematic condition whereby a single ultra-relativistic worldline intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points — forces a redefinition of entropy. In standard thermodynamics, entropy is a single-sheet quantity that obeys the second law. When a worldline is non-injective, the same physical system appears simultaneously on N topological sheets. The entropy measured on any single sheet can decrease, as illustrated by a moving mirror that reflects incoherent thermal light into a coherent superposition. We show that this apparent decrease is compensated by entropy increases on the remaining sheets and in the environment required to maintain the non-injective trajectory. We propose the *Topological Entropy Principle*: for a closed system that includes all sheets and the environment, the topological entropy $S_{\text{top}} = \frac{1}{N}\sum_{i=1}^{N} S_i^{\text{total}}$ never decreases. This is not a new postulate but a consequence of the Topological Emergence Identity $N(\epsilon) \cdot \epsilon^{d-2} = O(1)$ established in the companion paper [1]. The principle unifies the arrow of time, the resolution of Maxwell's demon, and the generalised second law for black holes within the TPST–DGQ framework.

1 Introduction

The second law of thermodynamics states that the entropy of an isolated system never decreases.[7] This law has never been derived from first principles; it is a statistical observation. Thought experiments such as Maxwell's demon show that local entropy can apparently decrease, but only at the cost of increasing entropy elsewhere.

This paper presents a different mechanism for apparent local entropy decrease: worldline non-injectivity. For Lorentz factors $\gamma > \gamma_{\text{crit}}$, a single timelike worldline intersects a constant-time hypersurface Σ_t in $N > 1$ distinct spatial points. The same physical system appears simultaneously on N topological sheets. The entropy measured on one sheet can decrease while the entropy on the other sheets compensates, leaving the total topological entropy non-decreasing.

This phenomenon was uncovered in the Bricks Paradox [2] and shown to be necessary and sufficient for finite holographic spacetime in [1]. The quantum-mechanical consequences were derived in [3]. The present paper extends the framework to thermodynamics.

The paper is organised as follows. Section 2 reviews non-injectivity and the Extended Lorentz Transformations. Section 3 describes the relativistic mirror experiment and the apparent entropy decrease. Section 4 performs the complete entropy accounting, showing that the decrease on one sheet is compensated. Section 5 states the Topological Entropy Principle. Section 6 discusses consequences for the arrow of time, Maxwell's demon, and black hole entropy.

2 Worldline Non-Injectivity and Extended Lorentz Transformations

2.1 Kinematic origin of non-injectivity

Consider a timelike worldline $X^\mu(\tau)$ in Minkowski spacetime. In standard special relativity, the map $\tau \mapsto X^0(\tau)$ is strictly increasing, hence injective. For Lorentz factors $\gamma > \gamma_{\text{crit}}$, a new phenomenon appears: the same coordinate time t^* and spatial position $x = M$ can correspond to two distinct proper times $\tau_1 \neq \tau_2$:

$$\exists\, t^* \in \mathbb{R},\ \tau_1 \neq \tau_2 : \quad X^0(\tau_1) = X^0(\tau_2) = t^*, \quad X^1(\tau_1) = X^1(\tau_2) = M. \tag{1}$$

The worldline is then called *non-injective* with respect to the simultaneity foliation $\{\Sigma_t\}$.

2.2 The critical Lorentz factor

The transition from injective to non-injective behaviour occurs at γ_{crit}, which depends on the geometry of the worldline. In the Bricks Paradox [2], γ_{crit} is as low as 30 for a macroscopic system. In holographic settings, γ_{crit} scales with the inverse of the UV cutoff. For the purpose of the present work, we only need the existence of such a threshold. What matters is that for $\gamma > \gamma_{\text{crit}}$ the worldline is non-injective and the number of simultaneous appearances $N > 1$.

2.3 Extended Lorentz Transformations

When $N > 1$, the standard Lorentz boost must be replaced by a set of N Extended Lorentz Transformations (ELT):

$$x'_n = \gamma(x_n - vt) + \Phi_n, \quad \Phi_n = \gamma^2 v(\tau_n - \tau_1), \quad n = 1, \dots, N, \tag{2}$$

with $t'_n = \gamma(t - vx_n/c^2)$. For $N = 1$, $\Phi_1 = 0$ and the ELT reduces to the standard Lorentz boost. The phase offset Φ_n is the topological separation between the n-th sheet and the reference sheet.

2.4 Intersection multiplicity and UV scaling

In holographic settings, the number of intersections with a fixed-time hypersurface scales as:

$$N(\epsilon) \sim \frac{1}{\epsilon^{d-2}}, \tag{3}$$

where ϵ is the UV cutoff and d is the number of boundary spacetime dimensions. This scaling matches the degree of divergence of the Ryu–Takayanagi entanglement entropy [5] and is the key to the topological cancellation $N(\epsilon) \cdot \epsilon^{d-2} = O(1)$ established in [1].

2.5 Ontological Identity Principle

The N intersections do not represent N distinct physical entities. They are N appearances of a single entity whose identity is carried by the continuous worldline $X^\mu(\tau)$. Any operation applied to one sheet propagates coherently to all others. This principle guarantees that the topological average of physical quantities is well-defined.

3 The Relativistic Mirror Experiment

3.1 Setup

A train of proper length $2L$ moves with velocity v along the x-axis of the ground frame S, with $\gamma = (1 - v^2/c^2)^{-1/2} > \gamma_{\text{crit}}$. A mirror is fixed at the centre ($x' = 0$).

Two lightning strikes occur simultaneously at $t' = 0$ at the ends of the train:

$$E_1 : \quad x' = +L, \; t' = 0 \quad \text{(front)}, \tag{4}$$

$$E_2 : \quad x' = -L, \; t' = 0 \quad \text{(rear)}. \tag{5}$$

In the ground frame S:

$$E_1 : \quad t_1 = \frac{\gamma v L}{c^2}, \; x_1 = \gamma L, \tag{6}$$

$$E_2 : \quad t_2 = -\frac{\gamma v L}{c^2}, \; x_2 = -\gamma L. \tag{7}$$

Thus E_2 lies in the past and E_1 in the future relative to the ground origin.

3.2 Reflection and coherence

Light from both strikes travels toward the mirror in S' and arrives simultaneously at $t'_r = L/c$. The reflection event is:

$$R: \quad x'_R = 0, \quad t'_R = L/c. \tag{8}$$

In S:

$$t_R = \gamma L/c, \quad x_R = \gamma v L/c. \tag{9}$$

The reflected light arrives at the ground observer at:

$$T_{\text{refl}} = t_R + x_R/c = \gamma L(1/c + v/c^2). \tag{10}$$

Now suppose the light from the strikes is *incoherent thermal radiation* — a mixed state with high von Neumann entropy. In S', the two wave packets arrive at the mirror simultaneously and are reflected as a coherent superposition:

$$|\Psi_R\rangle = \frac{1}{\sqrt{2}}\big(|E_1\rangle + |E_2\rangle\big). \tag{11}$$

The von Neumann entropy of this pure state is zero, whereas the incident thermal radiation was a mixed state with entropy $S_{\text{inc}} > 0$.

3.3 Single-sheet entropy decrease

The von Neumann entropy on the ground observer's sheet is:

$$S_{\text{sheet}} := -\operatorname{Tr}(\rho_{\text{sheet}} \log \rho_{\text{sheet}}). \tag{12}$$

Before the reflection, the radiation on the ground sheet is a mixed state with $S_{\text{sheet}} = S_{\text{inc}} > 0$. After the reflection, the radiation is in the pure state (11), so $S_{\text{sheet}} = 0$.

The single-sheet entropy decreases from S_{inc} to 0. If this were the only relevant quantity, the second law would be violated.

4 Complete Entropy Accounting

The single-sheet decrease identified in Section 3.3 is not a violation of the second law. It is compensated by two contributions that are invisible from the perspective of a single sheet.

4.1 Contribution 1: Entropy cost of non-injectivity

The non-injective regime $\gamma > \gamma_{\text{crit}}$ requires the train to be accelerated. This acceleration requires an energy input E_{acc} that is dissipated as heat in the environment, increasing the entropy of the environment by at least:

$$\Delta S_{\text{env}} \geq \frac{E_{\text{acc}}}{T}, \tag{13}$$

where T is the environmental temperature. The energy E_{acc} scales with γ and hence grows rapidly for $\gamma > \gamma_{\text{crit}}$.

This contribution is the direct relativistic analogue of the Landauer principle in Maxwell's demon[8, 9]: the cost of preparing the coherent configuration is paid by the environment.

4.2 Contribution 2: Entropy on other sheets

The Ontological Identity Principle states that the N sheets are not independent. The information about the incoherent incident radiation is not destroyed by the reflection; it is redistributed across the N sheets of the non-injective worldline.

Define the entropy on the i-th sheet as $S_i = -\operatorname{Tr}(\rho_i \log \rho_i)$, where ρ_i is the density matrix of the radiation on the i-th sheet. For the reflected coherent state, the sheet corresponding to the ground observer has $S_1 = 0$. The information about the original mixed state is stored on the remaining $N-1$ sheets, so:

$$\sum_{i=2}^{N} S_i \geq S_{\text{inc}}. \tag{14}$$

This inequality holds because the reflection produces a pure state only on the observer's sheet. The remaining $N-1$ sheets retain the correlations of the original mixed state, encoded in the inter-sheet entanglement established by the Ontological Identity Principle. The information of the original thermal radiation is redistributed across sheets, not destroyed. The total entropy over all sheets is:

$$S_{\text{total}}^{\text{rad}} = \sum_{i=1}^{N} S_i = S_1 + \sum_{i=2}^{N} S_i = 0 + \sum_{i=2}^{N} S_i \geq S_{\text{inc}}. \tag{15}$$

The total radiation entropy does not decrease.

4.3 The complete entropy balance

The total entropy change of the closed system (radiation on all sheets + environment) is:

$$\Delta S_{\text{closed}} = \Delta S_{\text{total}}^{\text{rad}} + \Delta S_{\text{env}} = \left(\sum_{i=1}^{N} S_i - S_{\text{inc}} \right) + \frac{E_{\text{acc}}}{T} \geq 0. \tag{16}$$

The inequality follows from (14) and (13).

The apparent entropy decrease on the ground observer's sheet is real and measurable. It is not a statistical fluctuation. It is a genuine consequence of non-injectivity. But it is compensated by entropy stored on the other sheets and in the environment, leaving the second law intact for the complete system.

4.4 Connection to the topological cancellation

The exact compensation in (16) is guaranteed by the same algebraic identity that regularises UV divergences:

$$N(\epsilon) \cdot \epsilon^{d-2} = O(1). \tag{17}$$

The multiplicity N that distributes the entropy across sheets is the same multiplicity that cancels the UV divergence of the RT area and the Coulomb self-energy. Entropy conservation across sheets is not an additional postulate; it is a consequence of the Topological Emergence Identity.

5 The Topological Entropy Principle

5.1 Definition of topological entropy

Definition 5.1 (Topological entropy)**.** *For a physical system described by a non-injective worldline with intersection multiplicity $N(\epsilon)$, the* topological entropy *is:*

$$S_{\text{top}} := \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} S_i^{\text{total}}, \tag{18}$$

where S_i^{total} is the total entropy on the i-th sheet, including the entropy of the system, the apparatus, and all relevant environmental degrees of freedom that have interacted with the system.

Remark 5.2. *The inclusion of all relevant degrees of freedom in S_i^{total} is essential. A single-sheet entropy that includes only the radiation (as in the mirror experiment before the full accounting of Section 4) can decrease. The topological entropy defined here includes* all *entropy contributions on each sheet, making it a closed-system quantity.*

5.2 Statement of the principle

Theorem 5.3 (Topological Entropy Principle)**.** *For any closed physical system described by a non-injective worldline with intersection multiplicity $N(\epsilon) \sim \epsilon^{-(d-2)}$, the topological entropy defined in* (18) *is non-decreasing in the direction of increasing coordinate time t:*

$$\frac{d}{dt} S_{\text{top}} \geq 0. \tag{19}$$

Single-sheet entropies S_i may increase or decrease. The topological average never decreases.

Proof sketch. The proof proceeds in three steps.

Step 1: Subadditivity and information conservation. By the strong subadditivity of von Neumann entropy[10], the total entropy of the N-sheet system satisfies:

$$\sum_{i=1}^{N} S_i^{\text{total}} \geq S_{\text{global}}, \tag{20}$$

where S_{global} is the entropy of the global N-sheet state. Under unitary evolution, S_{global} is conserved. Any decrease of S_i^{total} on one sheet must therefore be compensated by an increase on other sheets or in the environment, by strong subadditivity. Environmental irreversibility contributes an additional non-negative term, giving:

$$\sum_{i=1}^{N} S_i^{\text{total}} = \text{const} + \Delta S_{\text{env}}, \quad \Delta S_{\text{env}} \geq 0. \tag{21}$$

Step 2: Non-decrease of the average. Dividing (21) by N:

$$S_{\text{top}}(t_2) - S_{\text{top}}(t_1) = \frac{\Delta S_{\text{env}}}{N} \geq 0. \tag{22}$$

Step 3: Finiteness guaranteed by the TEI. The topological average S_{top} is finite and well-defined by the same Topological Emergence Identity $N(\epsilon) \cdot \epsilon^{d-2} = O(1)$ that ensures finiteness of the RT entropy and the Coulomb self-energy in the companion paper [1]. This guarantees that S_{top} does not diverge as $\epsilon \to 0$. □

Remark 5.4. *The standard second law, $\Delta S \geq 0$ for an isolated system, is recovered as the $N = 1$ special case of the Topological Entropy Principle. For $N = 1$, the worldline is injective, $S_{\rm top} = S_1^{\rm total}$, and the principle reduces to the classical second law.*

6 Consequences

6.1 Arrow of time

The direction of time is associated with the non-decrease of $S_{\rm top}$. In the companion paper [1], the arrow of time was shown to be the direction in which the worldline multiplicity $N(\epsilon)$ increases. These two descriptions are consistent: N increases as the worldline deepens its non-injectivity, and the Topological Entropy Principle ensures that $S_{\rm top}$ cannot decrease as N grows.

The birth of spacetime — the transition $N : 1 \to N > 1$ at $\gamma = \gamma_{\rm crit}$ — corresponds to the origin of the thermodynamic arrow of time. Before this transition, $N = 1$ and the system is in a quasi-classical regime with standard entropy. After the transition, $N > 1$ and the topological entropy structure emerges.

6.2 Resolution of Maxwell's demon

A Maxwell demon operating a relativistic mirror could, in principle, use single-sheet entropy decrease to sort particles and violate the second law on one sheet. However, the demon's own physical apparatus must operate at $\gamma > \gamma_{\rm crit}$, requiring energy input $E_{\rm acc}$ and generating $\Delta S_{\rm env} \geq E_{\rm acc}/T$. Moreover, the demon's memory — the record of which sheet it operates on — is distributed across all N sheets by the Ontological Identity Principle. When the demon attempts to exploit the single-sheet entropy decrease, it necessarily increases the entropy on the other sheets.

The Topological Entropy Principle guarantees that no demon operating within a non-injective worldline framework can decrease the total topological entropy. The resolution does not require invoking a thermodynamic cost of measurement separately; it is automatically included in the entropy balance of Section 4.

6.3 Black hole entropy and the generalised second law

In the TPST framework [1], the regularised holographic entropy of a black hole is the topological average:

$$S_{\rm BH}^{\rm top} = \frac{1}{N(\epsilon)} \sum_{i=1}^{N(\epsilon)} \frac{\mathrm{Area}(\gamma_{A,i})}{4G_N} = O(1). \tag{23}$$

This is finite by the topological cancellation of Lemma 3 of [1].

The Bekenstein–Hawking entropy [6] $S_{\rm BH} = A/(4G_N)$ is the single-sheet ($N = 1$) special case of (23). In the multi-sheet regime, the black hole entropy is an emergent, sheet-dependent quantity.

The generalised second law (GSL)[12] states that the total entropy of a black hole plus its exterior matter never decreases. In the present framework, the GSL is a direct consequence of the Topological Entropy Principle applied to the system comprising the black hole, the Hawking radiation[11], and all environmental degrees of freedom:

$$\Delta S_{\rm top}^{\rm total} = \Delta S_{\rm top}^{\rm BH} + \Delta S_{\rm top}^{\rm Hawking} + \Delta S_{\rm top}^{\rm env} \geq 0. \tag{24}$$

The apparent decrease of $S_{\rm BH}$ during Hawking evaporation — the black hole loses mass and hence area — is compensated by the increase of $S_{\rm top}^{\rm Hawking}$ on the other sheets. This provides a topological resolution of the black hole information paradox without invoking firewalls or non-unitary evolution: the information is not destroyed but redistributed across the N topological sheets. The Page curve — the time-evolution of the entanglement entropy of Hawking radiation — is recovered as the single-sheet projection of the conserved topological entropy.

7 Conclusions

We have shown that worldline non-injectivity forces a redefinition of entropy. When a worldline is non-injective, the same physical system appears on N topological sheets. The entropy on a single sheet can decrease, as demonstrated by the relativistic mirror experiment. This decrease is not a violation of the second law but a consequence of the redistribution of entropy across sheets and to the environment.

The Topological Entropy Principle — $dS_{\rm top}/dt \geq 0$ for the complete closed system over all sheets — is the correct statement of the second law in the multi-sheet regime. It reduces to the classical second law for $N = 1$.

Three consequences follow without additional postulates.

First, the arrow of time is the direction of increasing N and increasing $S_{\rm top}$, consistent with the companion result of [1].

Second, Maxwell's demon cannot exploit single-sheet entropy decrease to violate the total topological entropy non-decrease, because the demon's apparatus and memory are themselves distributed across all N sheets.

Third, the generalised second law for black holes follows directly: the apparent decrease of Bekenstein–Hawking entropy during Hawking evaporation is compensated by entropy on the other sheets, and the information is redistributed rather than destroyed.

The central identity remains:

$$N(\epsilon) \cdot \epsilon^{d-2} = O(1), \tag{25}$$

which guarantees that the topological entropy is finite and that the second law holds at every level of physical theory — holographic, electromagnetic, quantum mechanical, and thermodynamic.

Declarations

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